Geometric group theory
Problem sheet 2

Lent 2015

1. Prove that the free group F_2 contains a subgroup isomorphic to F_∞ (where ∞ is countable).

2. Let Γ be a group with a finite generating set S. The *growth function* of the pair (Γ, S) is defined to be
$$\beta_{\Gamma,S}(n) = \#\{\gamma \in \Gamma \mid l_S(\gamma) \leq n\} \ .$$
   (a) State and prove a quasi-isometry invariance result for growth functions.
   (b) Compute (coarsely) the growth function of $\mathbb{Z}^r$ for $r \in \mathbb{N}$.
   (c) Deduce that $\mathbb{Z}^r$ is quasi-isometric to $\mathbb{Z}^s$ if and only if $r = s$.

3. Let $\Gamma = \langle S \mid R\rangle$. The *conjugacy problem* asks for an algorithm that takes as input a pair of words $u, v \in S^*$ and determines whether or not the elements they represent in Γ are conjugate. Show that the conjugacy problem is solvable in the free group F_r (for $r < \infty$).

4. A group Γ is called *residually finite* if, for every $\gamma \in \Gamma \smallsetminus 1$, there exists a homomorphism to a finite group $f : \Gamma \to Q$ such that $f(\gamma) \neq 1$. Prove that if Γ is finitely presented and residually finite then the word problem is solvable in Γ.

5. For each positive integer k, find a presentation of $\mathbb{Z}$ with Dehn function $\delta(n) = kn$.

6. A *retraction* is a homomorphism of groups $r : G \to H$ with a right-inverse homomorphism $i : H \to G$ (i.e. $r \circ i = \mathrm{id}_H$). In this case, H is called a *retract* of G.
   (a) Prove that if G is finitely presentable then H is also finitely presentable.

(b) Prove that $\delta_H \preceq \delta_G$.

(c) For finitely presented groups Γ_1, Γ_2, show that

$$\delta_{\Gamma_1 * \Gamma_2}(n) \simeq \max\{\delta_{\Gamma_1}(n), \delta_{\Gamma_2}(n)\} \ .$$

(The group $\Gamma_1 * \Gamma_2$ is the *free product* of Γ_1 and Γ_2. You may wish to use the *normal form theorem for free products*, which asserts that if $a_i \in \Gamma_1 \smallsetminus 1$ and $b_i \in \Gamma_2 \smallsetminus 1$ then the product $\prod_i a_i b_i \neq 1$.)

Addendum: Part (c) turns out to be equivalent to an open problem! However, you can do the question if you assume that δ_{Γ_1} and δ_{Γ_2} are *super-additive*, meaning that $\delta_{\Gamma_i}(m+n) \geq \delta_{\Gamma_i}(m) + \delta_{\Gamma_i}(n)$.

7. A presentation $\mathcal{P} \equiv \langle S \mid R \rangle$ is called *Dehn* if it has the following property: in every van Kampen diagram D over $\mathcal{P}$ of positive area, there exists a 2-cell e so that $\partial e \cap \partial D$ contains an arc of length greater than half the length of ∂e. Show that if $\mathcal{P}$ is Dehn then $\delta_{\mathcal{P}}$ is linear.

8. Compute $\delta_{\mathbb{Z}^r}$ up to coarse equivalence.

9. Consider the finitely presented group $B = \langle a, b \mid aba^{-1}b^{-2} \rangle$. For each n, exhibit a van Kampen diagram for the word

$$w_n = a^n b a^{-n} b a^n b^{-1} a^{-n} b^{-1} \ .$$

10. The purpose of this question is to prove that a group Γ that acts properly discontinuously and cocompactly *but perhaps not freely* on a simply connected 2-complex $\widetilde{X}$ is finitely presented.

    (a) Show that we may assume that, if a cell of $\widetilde{X}$ is fixed set-wise by an element $\gamma \in \Gamma$, then it is fixed point-wise.

    (b) Let $\mathcal{F}$ be the set of finite subgroups of Γ that stabilize a vertex in $\widetilde{X}$. Prove that $\mathcal{F}$ is a finite union of conjugacy classes of subgroups.

    (c) Exhibit a compact, simply connected space $\widetilde{Z}$ such that every subgroup in $\mathcal{F}$ admits a free action on $\widetilde{Z}$.

    (d) Describe a locally finite, simply connected cell complex $\widetilde{Y}$ together with a free, cocompact Γ-action and a Γ-equivariant map $\widetilde{Y} \to \widetilde{X}$ such that every point pre-image is homeomorphic to $\widetilde{Z}$.